

Energy of a charged sphere

A total charge Q is distributed uniformly into a spherical volume of radius R . Find the electrostatic energy of this configuration by each of the following three methods:

- a) Evaluate the work done to build up the charged sphere “layer after layer” by carrying the requiring amount of charge from infinite distance.
- b) Evaluate the volume integral of $u_E = \epsilon_0 |\mathbf{E}|^2 / 2$, where \mathbf{E} is the electric field.
- c) Evaluate the volume integral of $\rho V / 2$, where ρ is the charge density and V is the electrostatic potential. Discuss the differences with the approach used in **b**).

Solution

a) We build up the sphere by adding subsequent infinitesimal layers of charge (carried from infinite distance). From Gauss's theorem we know that, for an uniformly charged sphere having charge density ρ , radius r , and total charge $q = q(r) = \rho(4\pi r^3/3)$, the field and the potential outside the sphere are those of a point charge q located in the center. Thus, in building the sphere, when a new layer of charge $dq = \rho 4\pi r^2 dr$ is added, its charge will be located at the potential $V(r) = k_0 q(r)/r$, where $k_0 = 1/4\pi\epsilon_0$; thus, the corresponding stored amount of electrostatic energy is $V(r)dq$. Integrating over r up to the final radius R we find for the total energy U_0

$$\begin{aligned} U_0 &= \int_0^R V(r)dq = \int_0^R k_0 \frac{q(r)}{r} \rho 4\pi r^2 dr \\ &= \int_0^R k_0 \rho \left(\frac{4\pi}{3} r^3 \right) \frac{1}{r} \rho 4\pi r^2 dr = \frac{4\pi \rho^2}{3\epsilon_0} \int_0^R r^4 dr = \frac{4\pi \rho^2 R^5}{15\epsilon_0} \\ &= \frac{3k_0 Q^2}{5 R}, \end{aligned} \quad (1)$$

being $\rho = Q/(4\pi R^3/3)$.

b) The electric field is easily found from Gauss's theorem to be

$$E(r) = \frac{Q}{4\pi\epsilon_0} \times \begin{cases} r/R^3 & (r < R) \\ 1/r^2 & (r > R) \end{cases} \quad (2)$$

The two integrals we need are

$$\int_0^R \left(\frac{r}{R^3} \right)^2 4\pi r^2 dr = \frac{4\pi}{5R}, \quad \int_R^\infty \left(\frac{1}{r^2} \right)^2 4\pi r^2 dr = \frac{4\pi}{R}, \quad (3)$$

so that we obtain

$$U_0 = \int_0^\infty \frac{\epsilon_0}{2} E^2(r) 4\pi r^2 dr = \frac{\epsilon_0}{2} \left(\frac{Q}{4\pi\epsilon_0} \right)^2 \left(\frac{4\pi}{5R} + \frac{4\pi}{R} \right) = \frac{3k_0 Q^2}{5 R}. \quad (4)$$

c) The electrostatic potential is given by

$$V(r) = \frac{Q}{4\pi\epsilon_0} \times \begin{cases} (-r^2/2R^3 + 3/2R) & (r < R) \\ 1/r & (r > R) \end{cases} \quad (5)$$

Since $\rho = 0$ for $r > R$, we only need to compute the integral of $\rho V/2$ from 0 to R :

$$\begin{aligned} U_0 &= \frac{1}{2} \int_0^R \frac{Q}{4\pi\epsilon_0} \rho \left(-\frac{r^2}{2R^3} + \frac{3}{2R} \right) 4\pi r^2 dr = \frac{Q}{4\epsilon_0} \frac{Q}{4\pi R^3/3} \left(-\frac{R^2}{5} + R^2 \right) \\ &= \frac{3}{4} \frac{Q^2}{4\pi\epsilon_0 R^3} \frac{4R^2}{5} = \frac{3k_0 Q^2}{5 R}. \end{aligned} \quad (6)$$

It is noticeable that, while methods **b)** and **c)** both give the correct result (as they should), they remind us that it is incorrect to give to the “energy density” of the electric field a physical interpretation as “energy contained in a certain region of space”. In fact, if we identify the energy density with $\epsilon_0 \mathbf{E}^2/2$ as in **b)**, we would conclude that the energy spreads out over the whole space, up to infinity; if, on the other hand, we take ρV as the energy density as in **c)**, it is “contained” only in the volume of the sphere, i.e. “where the charge is”. Thus, the concept of energy density is ambiguous, while the total electrostatic energy of the system is well defined.